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LETTER TO THE EDITOR

Fake Airy functions and the asymptotics of reflectionlessness

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Abstract. Two classes of analytic refractive-index profile $P^2(z, \varepsilon)$, whose reflection coefficients r are zero for all values of a parameter ε, are studied as $\varepsilon \to 0$. The aim is to understand why $r=0$ rather than $r \propto \exp(-1/\varepsilon)$ as for generic profiles. We find that reflectionlessness is a consequence of the fact that transition points of P^2 (zeros or poles in the complex z plane) form tight clusters (whose size vanishes with ε) which can be regarded neither as coalesced nor well separated. Expansion near a cluster yields the local wave not as the usual Airy function, whose Stokes phenomenon generates reflection, but as Bessel functions of half-integer order (fake Airy functions) which are exactly trigonometric functions with no Stokes phenomenon and so no reflection.

Wave reflections, described by the one-dimensional Helmholtz equation

$$\frac{\mathrm{d}^2\psi(z)}{\mathrm{d}z^2} + \frac{1}{\varepsilon^2} P^2(z, \varepsilon)\psi(z) = 0 \tag{1}$$

with the refractive index profile P positive real on the real z axis and analytic in a strip including the real axis, are exponentially weak in the short-wave limit $\varepsilon \to 0$. The reflections arise from transition points (zeros or poles of P^2) in the complex z plane, and can be calculated using the phase-integral method (Heading 1962, Fröman and Fröman 1965, Berry and Mount 1972). In the analogous problem where z represents time and ψ the coordinate of a classical harmonic oscillator whose frequency P is altered, $\varepsilon \to 0$ is the limit of slow change, and the analogue of exponentially weak reflection is exponentially small change in the adiabatic 'invariant'.

It is, however, easy to construct profiles for which the reflection is identically zero for all ε. In the oscillator analogue, the invariant is exactly conserved and in the quantum generalisation the states are the same in the infinite past and future. A natural question, which we answer here, is: how does reflectionlessness show itself in the small-ε asymptotics?

A large class of reflectionless profiles can be constructed (see e.g. Berry 1987) simply by defining the solution as a wave propagating purely in one direction. Thus

$$\psi(z) \equiv P_0^{-1/2}(z) \exp\left\{\frac{\mathrm{i}}{\varepsilon} \int_0^z \mathrm{d}z'\, P_0(z')\right\} \tag{2}$$

where the wavenumber P_0 is positive real on the real axis and analytic near it. This satisfies (1) with profile

$$P^2(z, \varepsilon) = P_0^2(z) - \varepsilon^2 P_0^{1/2}(z) \frac{\mathrm{d}^2}{\mathrm{d}z^2} P_0^{-1/2}(z). \tag{3}$$

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It is tempting but wrong to argue that when ε is small P can be approximated by P_0 and the short-wave reflection amplitude r then obtained by the phase-integral method. This would give the result that although r vanishes to all orders in ε it has a non-zero value determined by the transition point z^* closest to the real axis in the upper half-plane. Let z^* be an Mth-order zero of P_0^2 ($M<0$ corresponds to a pole). Then phase-integral analysis (e.g. Fröman and Fröman 1965) gives an asymptotic solution where in addition to (2) there is a reflected wave with amplitude

$$r \approx -2\mathrm{i}\cos\left\{\frac{\pi}{M+2}\right\}\exp\left\{-\frac{2}{\varepsilon}\,\mathrm{Im}\int_{x_0}^{z^*} P_0(z)\,\mathrm{d}z\right\}. \tag{4}$$

One way to obtain this result is to make a local expansion of the wave equation near z^* (with P_0 replacing P). This would give

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}+\frac{A(z-z^*)^M}{\varepsilon^2}\psi=0 \tag{5}$$

whose solutions are the Bessel functions (Abramowitz and Stegun 1964)

$$\psi(z)\approx(z-z^*)^{1/2}J_{[\pm 1/(M+2)]}\left(\frac{2A^{1/2}}{\varepsilon(M+2)}(z-z^*)^{1+M/2}\right). \tag{6}$$

Continuation formulae between $\mathrm{Re}(z-z^*)>0$ and $\mathrm{Re}(z-z^*)<0$ then lead directly to (4).

In this wrong argument, the reflection has been generated by the Stokes phenomenon (Stokes 1864, Dingle 1973): in the asymptotics of these Bessel functions, small exponentials are born across 'Stokes lines' emanating from z^*. For the most familiar case $M=1$, the appropriate combination of $J_{\pm 1/3}$ is the rainbow integral of Airy (1838).

Of course, the foregoing analysis has to be wrong, because the profile (3) generates the exact solution (2) which has no reflected wave. The error lies in replacing P by P_0, because the difference, although containing the small coefficient ε^2, is singular at the transition point z^* of P_0 and so cannot be neglected there. Local expansion near z^* gives

$$P^2(z,\varepsilon)\approx A(z-z^*)^M-\frac{\varepsilon^2 M(M+4)}{16(z-z^*)^2}. \tag{7}$$

Thus we see that in P^2 the effect of the 'perturbation' is to split the isolated Mth-order zero of P_0^2 into a double pole at z^* surrounded at a distance of order $\varepsilon^{2/(M+2)}$ by $M+2$ equally spaced simple zeros. Although the size of this cluster vanishes with ε, it is wrong, as we have seen, to treat it as a single collapsed transition point of order M. Nor is it correct to treat each transition point in the cluster as if it were well separated from all the others, because that would be justified only if the integral of P/ε between any two zeros is large, whereas in fact this quantity is of order unity (the integral between a zero and the double pole diverges).

The cluster of $M+2$ transition points must therefore be treated as a whole. This can be achieved by solving (1) exactly with P^2 replaced by its local approximation (7). The solution is

$$\psi(z)\approx(z-z^*)^{1/2}J_{\pm 1/2}\left(\frac{2A^{1/2}}{\varepsilon(M+2)}(z-z^*)^{1+M/2}\right). \tag{8}$$

This differs from the wrong solution (6) only in that these Bessel functions have order $1/2$ instead of $1/(M+2)$. But this is a crucial difference, because $J_{\pm 1/2}$ are given exactly by trigonometric functions: their asymptotic expansions terminate at the first term and are valid in all angular sectors around z^*. In other words, there is no Stokes phenomenon for these functions.

For the simplest case $M = 1$, we have

$$\psi(z) \approx (z - z^*)^{-1/4} \cos\left\{\frac{2A^{1/2}}{3\varepsilon}(z - z^*)^{3/2} + \gamma\right\} \tag{9}$$

where γ is a constant. This is the 'fake Airy function', whose exact form is what would be obtained as the leading term of the asymptotic expansion of the true Airy function if the Stokes phenomenon were neglected. It is the miracle of reflectionlessness to replace true Airy functions by fake ones.

In asymptotics it is more common to be presented with equation (1) with P given, and seek a solution of the form (2). Again this leads to (3) but with P_0, rather than P, as the unknown. The phase-integral method gives a formal solution of (3) for P_0 as a series in powers of ε^2. Usually this series diverges because in the true solution the incident wave (2) generates a reflection, which could not be captured in a convergent power series. Here, however, the series terminates at its first term, because the first ε^2 phase-integral correction is cancelled by the ε^2 term already in the lowest-order term P^2.

A different class of reflectionless profile occurs in soliton theory (Dodd *et al* 1982). These have P^2 in (1) given by

$$P^2(z, \varepsilon) = 1 + m(m+1)\varepsilon^2 \operatorname{sech}^2 z. \tag{10}$$

Exact solutions are

$$\psi(z) = \prod_{n=1}^{m}\left(n \tanh z - \frac{\mathrm{d}}{\mathrm{d}z}\right) \exp\left\{\pm\frac{\mathrm{i}z}{\varepsilon}\right\}. \tag{11}$$

Here, as with the previous class (3) of profiles, reflectionlessness can be understood in the small-ε limit by analysing the appropriate cluster of transition points. In this case we have to expand P^2 near $\mathrm{i}\pi/2$:

$$P^2(z, \varepsilon) = 1 - \frac{m(m+1)\varepsilon^2}{(z - \mathrm{i}\pi/2)^2} \tag{12}$$

so the cluster is a double pole flanked by two simple zeros with separation $2\varepsilon\sqrt{m(m+1)}$.

As before, these transition points may be regarded neither as coincident nor well separated, and must be treated as a group. The exact solution of (1) with local approximate profile (12) is

$$\psi(z) \approx (z - \mathrm{i}\pi/2)^{1/2} J_{\pm(m+1/2)}[(z - \mathrm{i}\pi/2)/\varepsilon]. \tag{13}$$

Because the order is half-integer, these are again expressible exactly in terms of trigonometric functions, with no Stokes phenomenon and so no reflection.

Our asymptotic analysis provides another way to see what is special about these reflectionless profiles. In quantum mechanical terminology, not only must the potential have a special form but the values of energy, Planck's constant and the potential parameters must be tuned so as to manoeuvre the transition points into tight clusters with precisely specified internal structure. Only then are the local wavefunctions fake Airy functions (e.g. (8), (9) or (13)) rather than real ones.

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